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"A geometrical approach to the special stable distributions"

By L. J. SAVAGE

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The convolution formulas for the Cauchy distribution and for the distribution of the reciprocal of the square of a standard normal random variable are here derived from the geometry of circularly symmetric distributions on the plane. Other known derivations seem quite different, and in a sense, less elementary, though some are by no means less interesting. A few other illustrations of the geometrical method are also included.

Let a, a', b, and b' be points on a circle with $a \neq a'$ and $b \neq b'$, and suppose that the chords (a, a') and (b, b') intersect in a point c not exterior to the circle and not the same as a or b. Then the angle (a, c, b) is half the sum of the arcs (a, b) and (a', b'). This fact seems old and widely known. The special case in which $a' = b' = c$ is very familiar indeed (Euclid III 26 and 27), and the general case is easily inferred from the special one with the aid of the construction line (a, b').

If a and b are two points in the Euclidean plane, the <u>direction</u> from a to b is the vector from a to b normalized to unit length, and the <u>unoriented direction</u> is the unordered couple consisting of the direction and its negative, so that the unoriented direction from a to b is also the unoriented direction from b to a. The notion of a uniformly distributed random direction needs no explicit definition here, and a random unoriented direction will be called uniformly distributed if it has the same distribution as the unoriented direction associated with a uniformly distributed random direction. In this terminology, the geometric fact of the preceeding paragraph has the following probabilistic interpretation.

<u>Lemma</u> 1. If a random point P is uniformly distributed on the periphery of a circle in the plane and c is not exterior to that circle, then the unoriented direction from c to P is uniformly distributed.

For the present form of the lemma and of the argument leading to it, I thank E. J. G. Pitman and Paul Lévy, respectively.

Let Z be a random vector distributed with circular symmetry in the Euclidean vector plane. Assume, to avoid unimportant complications, that $\Pr(Z = 0) = 0$. Let X and Y be a pair of cartesian coordinates of Z , and let R and A be the corresponding polar coordinates, so that: $X = R \cos A$; $Y = R \sin A$; R and A are independent; $\Pr(R > 0) = 1$; and A is uniformly distributed on the reals modulo 2π .

For this note, interest centers on the example in which X and Y are independent and normally distributed with mean 0 and variance 1 , which will be called the standard example. The more general situation is treated mainly to emphasize the structure of the arguments.

Let "$\sim$" mean "is distributed like".

Theorem 1. For all real f and g , $fX + gY \sim (f^2 + g^2)^{1/2} X$.

Proof: $fX + gY = (f^2 + g^2)^{1/2} R \cos (A - \alpha)$, where $\alpha = \arctan f/g$. But $A - \alpha \sim A$ and is independent of R . Since $X = R \cos A$, this completes the proof.

Specialized to the standard example, Theorem 1 is the convolution formula for the normal distribution. The proof, however familiar, helps to set the stage.

Lemma 2. If σ and τ are nonnegative and not both 0 ,

$$\frac{XY}{(\sigma^2 X^2 + \tau^2 Y^2)^{1/2}} \sim \frac{X}{\sigma + \tau} .$$

Proof: Expressed in polar coordinates, the lemma says that R times a certain function of A is distributed like $R \cos A$ or, equivalently $R \sin A$. Since R and A are independent, it will be adequate to show that the function of A , namely

$$h(A) = \frac{(\sigma + \tau) \sin A \cos A}{(\sigma^2 \cos^2 A + \tau^2 \sin^2 A)^{1/2}} ,$$

is distributed like $\sin A$. But

$$h(A/2) = \frac{(\sigma + \tau)\sin A}{\{2(\sigma^2 + \tau^2) + 2(\sigma^2 - \tau^2)\cos A\}^{1/2}}$$

$$= \frac{\sin A}{\{1 + 2\rho\cos A + \rho^2\}^{1/2}},$$

where, $\rho = (\sigma - \tau)/(\sigma + \tau)$. Since $(\sin 2A, \cos 2A) \sim (\sin A, \cos A)$, what needs to be shown is that $h(A/2) \sim \sin A$ for each ρ in $[-1, 1]$.

Geometrically, if A is thought of as a random point P on the unit circle, then $h(A/2)$ is the sine of the angular coordinate B of that point as viewed from the point $c = (-\rho, 0)$. If the angular coordinate associated with the direction from c to P is B, that associated with the opposite direction is $B + \pi$, the sine of which is the negative of that of B. Therefore, $|\sin B|$ depends only on the unoriented direction from c to P. But according to Lemma 1, this unoriented direction is distributed just as it would be if B were uniformly distributed, that is, $|\sin B| \sim |\sin A|$. This, together with the remark that the distribution of $\sin A$ and $\sin B$ are both symmetric, completes the proof of the present lemma.

Corollary 1. If M and N are independent and normal with mean 0, then $L = MN/(M^2 + N^2)^{1/2}$ is normal with mean 0 and $s.d.(L) = (s.d.(M)^{-1} + s.d.(N)^{-1})^{-1}$.

This corollary has been proved (Shepp 1964) by recognizing its obvious equivalence to the standard-example application of the next theorem, which itself is obviously equivalent to the lemma.

Theorem 2. For nonnegative p and q,

$$\frac{p}{X^2} + \frac{q}{Y^2} \sim \frac{(p^{1/2} + q^{1/2})^2}{X^2}$$

Applied to the standard example, the theorem asserts that the distribution of the reciprocal of the square of a standard normal variable is stable of order $1/2$. Doetsch (1936) attributes the fact to Cesaro and gives two interesting, but not very probabilistic, demonstrations. Lévy (1940) independently discovered the fact in the course of a beautiful investigation of the Wiener process. For some further information, pursue "Stable distributions of order 1/2" in the Index of (Feller 1966).

The standard Cauchy distribution is the distribution of $C = Y/X = \tan A$.

<u>Theorem 3</u>. For all real f and g not both 0,

$$(fC + g)^{-1} \sim \frac{fC + g}{(f^2 + g^2)} .$$

Proof:

$$(fC + g)^{-1} = \frac{X}{fY + gX}$$

$$= \frac{f}{(f^2 + g^2)} \frac{(-gY + fX)}{(fY + gX)} + \frac{g}{f^2 + g^2}$$

$$\sim \frac{f}{(f^2 + g^2)} \frac{Y}{X} + \frac{g}{f^2 + g^2} ,$$

as in the proof of Theorem 1.

Theorem 3 has been proved (Menon 1962, p. 1270) by direct calculation with the Cauchy density.

<u>Theorem 4</u>. If $Z' = (X', Y')$ is independent of Z and it too is distributed with circular symmetry, then, for all real f and g,

$$f \frac{X'}{X} + g \frac{Y'}{Y} \sim (|f| + |g|) \frac{X'}{X} . \tag{1}$$

Proof: The left side of (1) is U/V, where

$$U = \frac{fYX' + gXY'}{(g^2X^2 + f^2Y^2)^{1/2}}$$

and

$$V = \frac{XY}{(g^2X^2 + f^2Y^2)^{1/2}} .$$

Given X and Y, U is distributed like X', as Theorem 1 shows. Therefore U is independent of V and distributed like X'. According to Lemma 2, $V \sim X/(|f| + |g|)$. This proves the theorem.

<u>Corollary</u> 2. If C and C' are independent and have the standard Cauchy distribution, then for all f and g, $fC + gC' \sim (|f| + |g|)C$.

This popular elementary fact has been demonstrated in several other ways. See for example (Feller 1966, p. 50). The proof given here is easily extended to the multivariate Cauchy distribution, that is, the distribution of a normally distributed vector divided by an independent, standard normally distributed number.

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If X and Y are independent and normal about 0, possibly with different variances, then $Z = XY/(X^2 + Y^2)^{1/2}$ is also normally distributed about 0. This is known and known to be tantamount to the stability of the distribution of X^{-2}. The present paper reduces the distribution of Z to a very elementary fact of plane geometry and shows geometrically that it also implies the stability of the Cauchy distribution.

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